

# CARDINALITY OF THE ELLIS SEMIGROUP ON COMPACT METRIC COUNTABLE SPACES

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**ABSTRACT.** Let  $E(X, f)$  be the Ellis semigroup of a dynamical system  $(X, f)$  where  $X$  is a compact metric space. We analyze the cardinality of  $E(X, f)$  for a compact countable metric space  $X$ . A characterization when  $E(X, f)$  and  $E(X, f)^* = E(X, f) \setminus \{f^n : n \in \mathbb{N}\}$  are both finite is given. We show that if the collection of all periods of the periodic points of  $(X, f)$  is infinite, then  $E(X, f)$  has size  $2^{\aleph_0}$ . It is also proved that if  $(X, f)$  has a point with a dense orbit and all elements of  $E(X, f)$  are continuous, then  $|E(X, f)| \leq |X|$ . For dynamical systems of the form  $(\omega^2 + 1, f)$ , we show that if there is a point with a dense orbit, then all elements of  $E(\omega^2 + 1, f)$  are continuous functions. We present several examples of dynamical systems which have a point with a dense orbit. Such systems provide examples where  $E(\omega^2 + 1, f)$  and  $\omega^2 + 1$  are homeomorphic but not algebraically homeomorphic, where  $\omega^2 + 1$  is taken with the usual ordinal addition as semigroup operation.

## 1. INTRODUCTION

We start the paper by fixing some standard notions and terminology. Our dynamical systems  $(X, f)$ 's will consist of a compact metric space  $X$  and a continuous function  $f : X \rightarrow X$ . The *orbit* of  $x$ , denoted by  $\mathcal{O}_f(x)$ , is the set  $\{f^n(x) : n \in \mathbb{N}\}$ , where  $f^n$  is  $f$  composed with itself  $n$ -times. A point  $x \in X$  is called a *periodic point* of  $f$  if there exists  $n \geq 1$  such that  $f^n(x) = x$ , and its *period* is  $s = \min\{n \in \mathbb{N} : f^n(x) = x\}$ . Let  $P_f$  be the set of all periods of the periodic points of  $(X, f)$ . A point  $x$  is called *eventually periodic* if its orbit is finite. The  $\omega$ -*limit set* of  $x \in X$ , denoted by  $\omega_f(x)$ , is the set of points  $y \in X$  for which there exists an increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that  $f^{n_k}(x) \rightarrow y$ . Observe that for each  $y \in \mathcal{O}_f(x)$ , we have that  $\omega_f(y) = \omega_f(x)$ . If  $\mathcal{O}_f(y)$  contains a periodic point  $x$ , then  $\omega_f(y) = \mathcal{O}_f(x)$ . We denote by  $\mathcal{N}(x)$  the collection of all neighborhoods of  $x \in X$ . The set of all accumulation points of  $X$  will be denoted by  $X'$ . For a successor ordinal  $\alpha = \beta + 1$ ,  $X^{(\alpha)} = (X^{(\beta)})'$  and for a limit ordinal

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$\alpha$ ,  $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ . The *Cantor-Bendixson rank* of  $x \in X$ , denoted by  $CB(x)$ , is the first ordinal  $\alpha < \omega_1$  such that  $x \in X^{(\alpha)}$  and  $x \notin X^{\alpha+1}$ . The *Cantor-Bendixson rank* of  $X$  is the first ordinal  $\alpha < \omega_1$  for which  $X^{(\alpha)} = \emptyset$ . The Stone-Čech compactification  $\beta(\mathbb{N})$  of  $\mathbb{N}$  with the discrete topology will be identified with the set of ultrafilters over  $\mathbb{N}$ . Its remainder is denoted by  $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$  is the set of all free ultrafilters on  $\mathbb{N}$ , where, as usual, each natural number  $n$  is identified with the fixed ultrafilter consisting of all subsets of  $\mathbb{N}$  containing  $n$ . For each  $A \subseteq \mathbb{N}$ ,  $A^*$  denotes the collection of all  $p \in \mathbb{N}^*$  such that  $A \in p$ . If  $A, B \in \mathcal{P}(\mathbb{N})$  and  $A \setminus B$  is finite, then we will write  $A \subseteq^* B$ , and  $A =^* B$  when  $A \subseteq^* B$  and  $B \subseteq^* A$ .

Given a discrete dynamical system  $(X, f)$  its *Ellis semigroup*, denoted  $E(X, f)$ , is defined as the pointwise closure of  $\{f^n : n \in \mathbb{N}\}$  in the compact space  $X^X$  with composition of functions as its algebraic operation. The Ellis semigroup is equipped with the topology inherited from the product space  $X^X$ . The Ellis semigroup of a discrete dynamical system was introduced by R. Ellis in [3] and has been very useful to study the topological behavior of the dynamical systems. The article [9] offers an excellent survey concerning applications of the Ellis semigroup. In the paper [7], the authors initiated the study of the continuity of the elements of  $E(X, f)^*$ . For instance, they point out that if  $X$  is a convergent sequence with its limit point, then all the elements of  $E(X, f)$  are either continuous or discontinuous. On the other hand, P. Szuca [14] showed that if  $X = [0, 1]$  and  $f^p$  is continuous for some  $p \in \mathbb{N}^*$ , then all the elements of  $E([0, 1], f)$  are continuous. In the same direction, some results are obtained in [5] when the phase space is the Cantor set. The main tool that have been used in all these investigations is the combinatorial properties of the ultrafilters on  $\mathbb{N}$ . Certainly, the Ellis semigroup can be described in terms of the notion of  $p$ -limits where  $p$  is an ultrafilter on the natural number  $\mathbb{N}$ . Indeed, given  $p \in \mathbb{N}^*$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in a space  $X$ , we say that a point  $x \in X$  is the  $p$ -limit point of the sequence, in symbols  $x = p - \lim_{n \rightarrow \infty} x_n$ , if for every neighborhood  $V$  of  $x$ ,  $\{n \in \mathbb{N} : f^n(x) \in V\} \in p$ . Observe that a point  $x \in X$  is an accumulation point of a countable set  $\{x_n : n \in \mathbb{N}\}$  of  $X$  iff there is  $p \in \mathbb{N}^*$  such that  $x = p - \lim_{n \rightarrow \infty} x_n$ .

The notion of a  $p$ -limit point has been used in several branches of mathematics (see for instance [2] and [4, p. 179]). A. Blass [1] and N. Hindman [11] formally established the connection between “the iteration in topological dynamics” and “the convergence with respect to an ultrafilter” by considering a more general iteration of the function  $f$  as follows: Let  $X$  be a compact space and  $f : X \rightarrow X$  a continuous function. For  $p \in \mathbb{N}^*$ , the  $p$ -iterate of  $f$  is the function  $f^p : X \rightarrow X$  defined by

$$f^p(x) = p - \lim_{n \rightarrow \infty} f^n(x),$$

for all  $x \in X$ . The description of the Ellis semigroup and its operation in terms of the  $p$ -iterates are the following:

$$E(X, f) = \{f^p : p \in \beta\mathbb{N}\}$$

$$f^p \circ f^q = f^{q+p} \text{ for each } p, q \in \beta\mathbb{N} \text{ (see [1], [11]).}$$

We will use the following notation

$$E(X, f)^* := E(X, f) \setminus \{f^n : n \in \mathbb{N}\}.$$

Besides, we have that  $\omega_f(x) = \{f^p(x) : p \in \mathbb{N}^*\}$  for each  $x \in X$ .

In this paper we are interested on the cardinality of the Ellis semigroup  $E(X, f)$ . The work of A. Köhler [10] and M. E. Glasner and Megrehisvili [8] contain very interesting results about when  $E(X, f)$  has cardinality at most  $2^{\aleph_0}$  (the so called *tame* dynamical systems). In [8] it is established the Bourgain-Fremlin-Talagrand dichotomy for dynamical systems: either  $|E(X, f)| \leq 2^{\aleph_0}$  or  $E(X, f)$  contains a copy of  $\beta\mathbb{N}$  and  $|E(X, f)| = 2^{2^{\aleph_0}}$ . We will be mostly concerned with the case when  $X$  is countable compact metrizable space. In this case, as the cardinality of  $E(X, f)$  is obviously bounded for the cardinality of  $X^X$ , then  $|E(X, f)| \leq 2^{\aleph_0}$ . Moreover, as  $E(X, f)$  is a separable metric space, the classical perfect set theorem says that  $E(X, f)$  is either (at most) countable or has cardinality  $2^{\aleph_0}$ . Thus a natural question is to determine conditions under which each of those alternatives hold. For instance, we characterize when  $E(X, f)$  and  $E(X, f)^*$  are finite. We prove that if the set of all periods of the periodic points of  $(X, f)$  is infinite, then  $|E(X, f)| = 2^{\aleph_0}$ . Concerning upper bounds, we prove that if  $(X, f)$  has a dense orbit and all elements of the Ellis semigroup are continuous, then  $|E(X, f)^*| \leq |X|$ . In the third section, it is shown that when  $(\omega^2 + 1, f)$  has a point with a dense orbit, then  $E(\omega^2 + 1, f)$  is countable and contains only continuous functions. An example of a continuous function  $f : \omega^3 + 1 \rightarrow \omega^3 + 1$  such that  $E(\omega^3 + 1, f)^*$  contains only discontinuous functions is given in section fourth. Additionally, we provide examples of dynamical systems with a dense orbit when the phase space is  $\omega^2 + 1$ . These dynamical systems also illustrate that  $E(\omega^2 + 1, f)$  and  $\omega^2 + 1$  can be homeomorphic but not algebraically homeomorphic (where  $\omega^2 + 1$  is equipped with the standard ordinal addition as semigroup operation).

## 2. CARDINALITY OF THE ELLIS SEMIGROUP

To start this section we state several auxiliary results that were proved in [6].

**Lemma 2.1.** *Let  $(X, f)$  be a dynamical system and  $x \in X$ .*

- (i) *Assume that  $x$  is periodic with period  $n$  and let  $l < n$ . Then,  $p \in (n\mathbb{N} + l)^*$  iff  $f^p(x) = f^l(x)$ .*

- (ii) Suppose that  $x$  is eventually periodic and that  $m \in \mathbb{N}$  is the smallest positive integer such that  $f^m(x)$  is a periodic point. If  $n$  is the period of  $f^m(x)$  and  $p \in (n\mathbb{N} + l)^*$  for some  $l < n$ , then  $f^p(x) = f^l(f^{nj}(x))$  where  $j = \min\{i : m \leq ni + l\}$ .
- (iii) Suppose that the orbit of  $x$  is infinite and  $\omega_f(x) = \mathcal{O}_f(y)$  for some periodic point  $y \in X$  with period  $n$ . If  $p, q \in (n\mathbb{N} + l)^*$  for some  $l < n$ , then  $f^p(x) = f^q(x)$ .
- (iv)  $f^p(f^n(x)) = f^n(f^p(x))$  for every  $n \in \mathbb{N}$ ,  $x \in X$  and every  $p \in \mathbb{N}^*$ .

The next theorem provides a necessary and sufficient conditions for  $E(X, f)$  to be finite, its elementary proof is omitted.

**Theorem 2.2.** *Let  $(X, f)$  be a dynamical system. Then  $E(X, f)$  is finite iff there exists  $M > 0$  such that  $|\mathcal{O}_f(x)| < M$  for each  $x \in X$ .*

Similarly, we will show a sufficient and necessary condition for  $E(X, f)^*$  to be finite.

**Theorem 2.3.** *Let  $(X, f)$  be a dynamical system.  $E(X, f)^*$  is finite iff there is  $M \in \mathbb{N}$  such that  $|\omega_f(x)| \leq M$  for each  $x \in X$ .*

*Proof.* Assume that  $E(X, f)^*$  is finite and let  $M = |E(X, f)^*|$ . Since  $\omega_f(x) = \{f^p(x) : p \in \mathbb{N}^*\}$ , we must have that  $|\omega_f(x)| \leq M$  for each  $x \in X$ .

Conversely, assume that there is  $M \in \mathbb{N}$  so that  $|\omega_f(x)| \leq M$  for each  $x \in X$ . Then, we know that every point of  $\omega_f(x)$  is periodic for each  $x \in X$  and so  $|P_f| \leq M$ . Hence, let  $P_f = \{b_1, \dots, b_n\}$  for some  $n \leq M$ . Define  $\phi : \mathbb{N}^* \rightarrow \prod_{i=1}^n \{0, \dots, b_i - 1\}$  by  $\phi(p) = (j_1, \dots, j_n)$  provided that  $p \in (b_i\mathbb{N} + j_i)^*$  for each  $1 \leq i \leq n$ , for every  $p \in \mathbb{N}^*$ . To see that  $E(X, f)^*$  is finite, it suffices to show that  $\phi(p) = \phi(q)$  iff  $f^p = f^q$ , for  $p, q \in \mathbb{N}^*$ . But this follows directly from clauses (i) and (iii) of Lemma 2.1.  $\square$

It is noteworthy that  $E(X, f)^*$  could be finite and  $E(X, f)$  could be infinite. For instance, if  $X$  is a convergent sequence with its limit point and  $f$  is the shift function, then  $E(X, f)$  is infinite and  $E(X, f)^*$  consists of only one point. The next corollary follows directly from Theorem 2.3.

**Corollary 2.4.** *Let  $(X, f)$  be a dynamical system. If  $E(X, f)^*$  is finite, then  $P_f$  is finite.*

The converse of the previous corollary is not true, we shall describe an example of a dynamical system  $(X, f)$  such that  $P_f$  is finite and  $E(X, f)^*$  infinite (Examples 4.3 and 4.4).

So far we have dealt only with conditions implying that  $E(X, f)$  or  $E(X, f)^*$  are finite. Now we show that  $E(X, f)$  has cardinality at least  $2^{\aleph_0}$ , whenever  $P_f$  is infinite. For that end we need the general form of the Chinese Remainder Theorem (for more properties about this theorem see for instance [13]).

**Lemma 2.5.** *The system of equations*

$$x \equiv r_i \pmod{m_i},$$

for  $i = 0, \dots, k$ , has an integer solution  $x$  iff  $\gcd(m_i, m_j)$  divides  $r_i - r_j$  for all  $i \neq j$ .

The following lemma is very important for the estimation of the cardinality of the Ellis semigroup by using ultrafilters on  $\mathbb{N}$ .

**Lemma 2.6.** *Let  $k \geq 1$  and  $m_1, \dots, m_k$  be positive integers,  $0 \leq r_i < m_i$ . Suppose that the following system  $E$  of equations has a solution:*

$$(1) \quad x \equiv r_i \pmod{m_i} \text{ for } i = 1, \dots, k.$$

*Then for every infinite  $A \subseteq \mathbb{N}$  there are an infinite subset  $B$  of  $A$  and positive integers  $s_1 < s_2$  such that the equation system  $E \cup \{x \equiv s_i \pmod{m} : i = 1, 2\}$  has a solution for all  $m \in B$  with  $m > s_2$ .*

*Proof.* By the Pigeon Hole Principle, there is an infinite  $B \subseteq A$  such that if  $m, m' \in B$ , then  $\gcd(m, m_i) = \gcd(m', m_i)$  for all  $1 \leq i \leq k$ . Observe that the number  $l_i = \gcd(m, m_i)$  does not depend on the choice of  $m \in B$ , for each  $1 \leq i \leq k$ . Consider the following system of equations:

$$(2) \quad x \equiv r_i \pmod{l_i} \text{ for } i = 1, \dots, k.$$

Since the system of equations (1) has a solution, by Lemma 2.5, we have that  $\gcd(m_i, m_j)$  divides  $r_i - r_j$  for every  $1 \leq i \leq k$ . As  $\gcd(l_i, l_j)$  divides  $\gcd(m_i, m_j)$ , then by Lemma 2.5 the system (2) has a solution. Choose two such solutions  $s_1 < s_2$ . If  $m \in B$  satisfies that  $s_2 < m$ , again by Lemma 2.5, then the system  $E \cup \{x \equiv s_i \pmod{m} : i = 1, 2\}$  has a solution.  $\square$

Next, we will see how the machinery of Number Theory and the ultrafilters on  $\mathbb{N}$  works.

**Theorem 2.7.** *Let  $(X, f)$  be a dynamical system. If  $P_f$  is infinite, then  $E(X, f)$  has no isolated points. In particular, if  $X$  is countable and  $P_f$  is infinite, then  $E(X, f)$  is homeomorphic to  $2^{\mathbb{N}}$ .*

*Proof.* First, we shall prove that  $E(X, f)$  has no isolated points. Let  $p \in \beta\mathbb{N}$  and  $V$  be an open set in  $X^X$  such that  $f^p \in V = \{g \in X^X : g(x_i) \in V_i \text{ for } i = 1, \dots, k\}$  where  $x_1, \dots, x_k \in X$  and  $V_1, \dots, V_k$  are open subsets of  $X$ . It suffices to show that there is  $q \in \beta\mathbb{N}$  such that  $f^q \neq f^p$  and  $f^q \in V$ . We need to consider two cases.

Case 1: There is  $1 \leq i \leq k$  such that  $x_i$  has infinite orbit. If  $B_j = \{n \in \mathbb{N} : f^n(x_j) \in V_j\}$ , then  $p \in B_j^*$  for every  $j \leq k$ . Hence, we can find  $n \in B_0 \cap B_1 \cap \dots \cap B_k$  so that  $f^n(x_i) \neq f^p(x_i)$ . Notice that  $f^n \in V$ .

Case 2:  $x_i$  has finite orbit for every  $1 \leq i \leq k$ . So each  $x_i$  must be eventually periodic of some period  $m_i$ . Pick  $r_i < m_i$  such that  $p \in (m_i\mathbb{N} + r_i)^*$  for each  $1 \leq i \leq k$ . By Lemma 2.1 (ii),  $f^q(x_i) = f^p(x_i)$  whenever  $q \in (m_i\mathbb{N} + r_i)^*$  for each  $1 \leq i \leq k$ . By Lemma 2.6, there are  $m \in P_f$

and  $s_1 < s_2 < m$  such that the system of equations  $x \equiv r_i \pmod{m_i}$  for  $i = 1, \dots, k$  together with the equation  $x \equiv s_j \pmod{m}$  has a solution for  $j = 1, 2$ . Pick  $p_j \in \bigcap_{i=1}^k (m_i \mathbb{N} + r_i)^* \cap (m \mathbb{N} + s_j)^*$  for  $j = 1, 2$ . Let  $y$  be a periodic point with period  $m$ . Then  $f^{p_1}(x_i) = f^{p_2}(x_i)$ , for every  $1 \leq i \leq k$ , and  $f^{p_j}(y) = f^{s_1}(y) \neq f^{s_2}(y) = f^{p_2}(y)$ . Therefore,  $f^{p_1}, f^{p_2} \in V$  and  $f^{p_1} \neq f^{p_2}$ .

If  $X$  is countable, then  $E(X, f)$  is homeomorphic to  $2^{\mathbb{N}}$  since it is compact metrizable without isolated points and zero dimensional  $\square$

Concerning the previous theorem, it would be also interesting to see whether the algebraic structure of  $E(X, f)$  is unique when  $X$  is countable and  $P_f$  is infinite. We also wonder about the existence of a dynamical system  $(X, f)$  for which  $P_f$  is finite and  $E(X, f)$  is uncountable.

**Corollary 2.8.** *For every countable ordinal  $\alpha \geq 1$ , there is a continuous function  $f : \omega^\alpha + 1 \rightarrow \omega^\alpha + 1$  such that  $E(\omega^\alpha + 1, f)$  is homeomorphic to the Cantor space  $2^{\mathbb{N}}$ .*

*Proof.* According to Theorem 2.7, it suffices to define a continuous function  $f : \omega^\alpha + 1 \rightarrow \omega^\alpha + 1$  for which  $P_f$  is infinite. For the case when  $\alpha = 1$ , it is very easy to define such a function. Suppose that  $\alpha > 1$ . Choose a subspace  $X$  of  $\omega^\alpha + 1$  so that  $\omega^\alpha + 1 = (\omega + 1) \oplus X$ . If  $f$  is the function defined for the case  $\alpha = 1$ , then we consider the function  $g = f \oplus Id : (\omega + 1) \oplus X \rightarrow (\omega + 1) \oplus X$ . Observe that  $g^p = f^p \oplus Id$ , for every  $p \in \beta(\mathbb{N})$ , which implies that  $E(\omega^\alpha + 1, g) \approx E(\omega + 1, f) \approx 2^{\mathbb{N}}$ .  $\square$

In the next theorem, we shall bound the cardinality of the Ellis semigroup for certain dynamical systems.

**Theorem 2.9.** *Let  $(X, f)$  be a dynamical system such that there is  $w \in X$  with a dense orbit. Suppose that  $f^p$  is continuous for every  $p \in \mathbb{N}^*$ . Then  $f^p = f^q$  iff  $f^p(w) = f^q(w)$ , for every  $p, q \in \mathbb{N}^*$ . In particular,  $|E(X, f)^*| \leq |X|$ .*

*Proof.* Let  $p \in \mathbb{N}^*$ . It suffices to prove that  $f^p$  is completely determined by its value at  $w$ . Fix  $x \in X$ . First, suppose that  $x$  is an isolated point. Then, there is  $n \in \mathbb{N}$  (depending on  $x$ ) such that  $f^n(w) = x$ . Thus,  $f^p(x) = f^p(f^n(w)) = f^n(f^p(w))$  for every  $p \in \mathbb{N}^*$ . Now, assume otherwise that  $x$  is a limit point. Choose a sequence  $(f^{m_k}(w))_{k \in \mathbb{N}}$  converging to  $x$ . Since  $f^p$  is continuous, we have that

$$f^p(x) = \lim_{k \rightarrow \infty} f^p(f^{m_k}(w)) = \lim_{k \rightarrow \infty} f^{m_k}(f^p(w)).$$

Therefore,  $f^p$  is completely determined by  $f^p(w)$ .  $\square$

## 3. COMPACT METRIC COUNTABLE SPACES

We remind the reader the classical result that every compact metric countable space is homeomorphic to a countable ordinal with the order topology (see [12]). In what follows, our dynamical systems will have  $\omega^\alpha + 1$  as a phase space, where  $\alpha$  is a countable ordinal with  $\alpha \geq 1$ . For our convenience,  $d$  will stand for the unique point of  $\omega^\alpha + 1$  with  $CB$ -rank equal to  $\alpha$ .

In the next, lemma we list some basic properties of the dynamical systems of the form  $(\omega^\alpha + 1, f)$  with a dense orbit.

**Lemma 3.1.** *Let  $(\omega^\alpha + 1, f)$  be a dynamical system with  $\alpha \geq 1$ , such that there exists  $w \in \omega^\alpha + 1$  with a dense orbit. Then, the following conditions hold:*

- (i)  $f(y)$  is a limit point for every  $y \in (\omega^\alpha + 1)'$ .
- (ii)  $w$  is isolated and its orbit consists of all isolated points of  $\omega^\alpha + 1$ .
- (iii) The range of  $f$  is  $\omega^\alpha + 1 \setminus \{w\}$ .
- (iv) If  $x \in (\omega^\alpha + 1)'$ , then  $\emptyset \neq f^{-1}(x) \subseteq (\omega^\alpha + 1)'$ .
- (v) If  $CB(z) = 1$ , then  $z$  is not periodic.

*Proof.* (i). Let  $y \in (\omega^\alpha + 1)'$  and suppose  $f(y)$  is isolated to get a contradiction. Since  $y$  is a limit point, then  $\{z \in \omega^\alpha + 1 : f(z) = f(y)\}$  is open and infinite. As the orbit of  $w$  is dense, there are  $k < l$  such that  $f(f^k(w)) = f(f^l(w)) = f(y)$ , thus the orbit of  $w$  is finite, which is a contradiction.

(ii). Since the orbit of  $w$  is dense, then it contains all isolated points of  $\omega^\alpha + 1$  and by (i) no limit point belongs to the orbit of  $w$ .

(iii). Since the orbit of  $w$  is dense, then it is clear that every limit point belongs to the range of  $f$  since it is compact. If  $f(y) = w$  for some  $y \in X$ , then  $y$  must be isolated and so  $w$  is periodic, which is impossible. So,  $w$  is not in the range of  $f$ .

(iv). Let  $x \in (\omega^\alpha + 1)'$ . By clause (iii), we have that  $f^{-1}(x) \neq \emptyset$ . Suppose that  $y \in f^{-1}(x)$  is isolated. Since the orbit of  $w$  is dense, there is  $n \in \mathbb{N}$  such that  $f^n(w) = y$  and thus  $f^{n+1}(w) = x$ . By (i),  $f^m(w)$  is a limit point for all  $m \geq n + 1$ , but this contradicts the fact that the orbit of  $w$  is dense.

(v) Suppose that  $z$  is a periodic point of period  $l$ . For each  $0 \leq j < l$ , fix disjoint clopen sets  $V_j$  such that  $V_j \cap \mathcal{O}_f(z) = \{f^j(z)\}$  and also  $z$  is the only limit point in  $V_0$  (since  $CB(z) = 1$ ). Moreover, we can also assume that  $f[V_{l-1}] \subseteq V_0$  and  $f[V_i] \subseteq V_{i+1}$  for  $0 < i < l - 1$  (notice that we cannot ask that  $f[V_0] \subseteq V_1$ ). Let  $V = V_0 \cup \dots \cup V_{l-1}$ . Since the orbit of  $w$  is dense, then there is  $j < l$  such that  $A = \{n \in \mathbb{N} : f^n(w) \in V_j \text{ and } f^{n+1}(w) \notin V\}$  is infinite. From the assumptions about the  $V_i$ 's, it is clear that  $j = 0$ . Let  $\{n_k : k \in \mathbb{N}\}$  be an infinite subset of  $A$  such that  $f^{n_k}(w) \rightarrow z$ . However,  $f(w) \in V$  but  $f(f^{n_k}(w)) \notin V$  for all  $k$ , contradicting the continuity of  $f$ . Consequently,  $z$  is not periodic.  $\square$

**Lemma 3.2.** *Let  $(\omega^\alpha + 1, f)$  be a dynamical system with  $\alpha \geq 1$  such that there exists  $w \in \omega^\alpha + 1$  with a dense orbit.*



- (i) Suppose  $x \in (\omega^\alpha + 1)'$  is such that  $CB(y) < CB(x) < \alpha$  for every  $y \in f^{-1}(x)$ . If  $(x_n)_{n \in \mathbb{N}}$  is a sequence such that  $x_n \rightarrow x$ ,  $CB(x_n) < CB(x)$ , for each  $n$ , and  $\sup\{CB(x_n) : n \in \mathbb{N}\} = CB(x)$ , then there is  $N \in \mathbb{N}$  such that  $CB(z) < CB(x_n)$  for every  $z \in f^{-1}(x_n)$  and  $n \geq N$ .
- (ii)  $f(d) = d$ .

*Proof.* (i) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence such that  $x_n \rightarrow x$ ,  $CB(x_n) < CB(x)$  for all  $n$  and  $CB(x_n)$  converges to  $CB(x)$ . We proceed by contradiction. By Lemma 3.1, and passing to a subsequence if necessary, there is a sequence  $(z_n)_{n \in \mathbb{N}}$  so that  $z_n \in f^{-1}(x_n)$  and  $CB(z_n) \geq CB(x_n)$  for each  $n \in \mathbb{N}$ . Choose a sequence  $(n_k)_{k \in \mathbb{N}}$  and  $y$  such that  $z_{n_k} \rightarrow y$ . Clearly  $y \in f^{-1}(x)$ . Since  $CB(z_{n_k}) \geq CB(x_{n_k})$ , then  $CB(y) \geq CB(x)$ , which is a contradiction.

(ii) Assume that  $f(d) = y$  for some  $y \in (\omega^\alpha + 1)' \setminus \{d\}$  with  $CB(y) < \alpha$ . Let  $d_n \rightarrow d$  such that  $CB(d_n) < \alpha$  for all  $n$  and  $\sup_n CB(d_n) = \alpha$ . By the continuity of  $f$  at  $d$ , we can assume without loss of generality that  $f(d_n) \neq d$  for all  $n$ . We claim that there is  $n \in \mathbb{N}$  such that  $CB(z) < CB(d_n)$  for all  $z \in f^{-1}(d_n)$ . Otherwise, for each  $n \in \mathbb{N}$ , there is  $z_n \in f^{-1}(d_n)$  such that  $CB(z_n) \geq CB(d_n)$ . Therefore, passing to a subsequence if it is necessary,  $z_n \rightarrow d$  and  $f(z_n) \rightarrow d$ , then  $f(d) = d$  which contradicts our assumption.

Fix  $n_0 \in \mathbb{N}$  such that  $CB(z) < CB(d_{n_0})$  for all  $z \in f^{-1}(d_{n_0})$ . Let  $y_1 = d_{n_0}$ . We can apply part (i) and obtain a point  $y_2$  such that  $CB(y_2) < CB(y_1)$  and  $CB(z) < CB(y_2)$  for every  $z \in f^{-1}(y_2)$ . This process has to end in a finite number of steps when we reach a point  $y_k$  such that  $CB(z) < CB(y_k) = 1$  for every  $z \in f^{-1}(y_k)$ . But this contradicts part (iv) of Lemma 3.1. Therefore,  $f(d) = d$ .  $\square$

**Theorem 3.3.** *Let  $(\omega^2 + 1, f)$  be a dynamical system such that there exists  $w \in \omega^2 + 1$  with a dense orbit. Then  $f^p$  is continuous, for every  $p \in \mathbb{N}^*$ , and  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$ .*

*Proof.* Fix  $p \in \mathbb{N}^*$ . Let  $\{d_n : n \in \mathbb{N}\}$  the set of all points of  $X$  with  $CB$ -rank equal to 1. According to Lemma 3.2, we know that  $f^p(d) = d$  and, by Lemma 3.1, we have that  $f(d_n)$  is a limit point for each  $n \in \mathbb{N}$ . Thus, we obtain that  $O_f(d_n) \subseteq \{d_m : m \in \mathbb{N}\} \cup \{d\}$  for each  $n \in \mathbb{N}$ . Besides, by Lemma 3.1 (v), we know that  $d_n$  cannot be periodic for all  $n \in \mathbb{N}$ .

First, we show that  $f^p(d_n) = d$  for each  $n \in \mathbb{N}$ . Fix  $n \in \mathbb{N}$ . To have this done we consider two cases: (a) Suppose that  $O_f(d_n)$  is finite. Then  $d_n$  is eventually periodic. Since  $d_m$  is not periodic for every  $m \in \mathbb{N}$ , then  $d$  belongs to the orbit of  $d_n$  and thus  $f^p(d_n) = d$ . (b) Suppose that  $O_f(d_n)$  is infinite. Since  $O_f(d_n) \subseteq \{d_m : m \in \mathbb{N}\} \cup \{d\}$ , then  $f^m(d_n) \rightarrow d$  and thus  $f^p(d_n) = d$ .

We are ready to prove the continuity of  $f^p$ . Let  $y \neq d$  be a limit point and  $(y_n)_{n \in \mathbb{N}}$  be a sequence of isolated points converging to  $y$ . We have already shown that  $f^p(y) = d$ . For each  $n \in \mathbb{N}$ , put  $y_n = f^{k_n}(w)$  for some  $k_n \in \mathbb{N}$ . Hence, we have that  $f^p(y_n) = f^{k_n}(f^p(w))$  for every  $n \in \mathbb{N}$ . Since  $f^p(w)$  is a



limit point, then  $f^p(y_n) \rightarrow d$ . Finally, if  $y_n \rightarrow d$ , then regardless of whether each  $y_n$  is isolated or not, we have that  $f^p(y_n) \rightarrow d$ .

Now we will verify that  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$ . For each  $n \in \mathbb{N}$ , let us fix a clopen subset  $V_n$  of  $\omega^2 + 1$  such that  $d_n$  is the only limit point in  $V_n$ . Consider the following sets:

$$B_n = \{i \in \mathbb{N} : f^i(w) \in V_n\}.$$

We claim that:

- (i) If  $p, q \in B_n^*$  for all  $n \in \mathbb{N}$ , then  $f^p = f^q$  and  $f^p(w) = d_n$ .
- (ii) If  $p, q \notin B_n^*$  for all  $n \in \mathbb{N}$ , then  $f^p = f^q =$  the constant function with constant value equal to  $d$ .

For every  $n \in \mathbb{N}$  fix  $p_n \in B_n^*$  and fix  $q \notin \bigcup_{n \in \mathbb{N}} B_n^*$ . Then we define  $F : \omega^2 + 1 \rightarrow E(X, f)$  as follows:

$$F(x) = \begin{cases} f^m & \text{if } x = f^m(w) \\ f^{p_n} & \text{if } x = d_n \text{ for some } n \in \mathbb{N} \\ f^q & \text{if } x = d. \end{cases}$$

It is not hard to prove that  $F$  is an homeomorphism.  $\square$

Theorem 3.3 is false for the space  $\omega^3 + 1$ . Certainly, in Example 4.4, we shall construct a continuous function  $f$  on  $\omega^3 + 1$  with a dense orbit for which  $f^p$  is discontinuous on  $\omega^3 + 1$ , for all  $p \in \mathbb{N}^*$ .

We end this section by making some remarks about the algebraic structure of  $E(\omega^2 + 1, f)$  when it has a dense orbit. It follows from the proof of the previous theorem that the semigroup operation of  $E(\omega^2 + 1, f)$  satisfies the following properties:

- (1) if  $p, q \in \mathbb{N}^*$ , then  $f^p \circ f^q =$  the constant function with constant value equal to  $d$ .
- (2) If  $p \in B_n^*$ , for some  $n \in \mathbb{N}$ , and  $k \in \mathbb{N}$ , then  $f^p \circ f^k(d_m) = f^k(f^p(d_m)) = f^k(d) = d$  for every  $m \in \mathbb{N}$ , and  $f^p \circ f^k(w) = f^k(d_n) = f^k(d_n)$ .
- (3) If  $p \notin \bigcup_{n \in \mathbb{N}} B_n^*$  and  $k \in \mathbb{N}$ , then  $f^p \circ f^k =$  the constant function with constant value  $d$ .

On the other hand, we observe that each ordinal of the form  $\omega^\alpha$  is a semigroup under the usual ordinal addition. We can extend this semigroup operation to  $\omega^\alpha + 1$  by simply declaring  $\omega^\alpha + \beta = \beta + \omega^\alpha = \omega^\alpha$  for all  $\beta \in \omega^\alpha + 1$ . Under this operation,  $\omega^\alpha + 1$  is a semigroup and the operation is left continuous. It is known that  $E(X, f)$  is also a left continuous semigroup for any dynamical system  $(X, f)$ . Considering a dynamical system  $(\omega^2 + 1, f)$  satisfying the conditions of Theorem 3.3, we can see that  $E(\omega^2 + 1, f)$  is not topological isomorphic (i. e., both isomorphic and homeomorphic) to  $\omega^2 + 1$  equipped with the above operation: Indeed, let  $\{d_n : n \in \mathbb{N}\}$  be the set of

limit points of  $\omega^2 + 1$  with  $CB$ -rank equal to 1. We have that, under ordinal addition,  $d_n + d_n < d$ , for every  $n \in \mathbb{N}$ , but  $f^p \circ f^q = \text{constant function with value } d$  provided that  $p, q \in \mathbb{N}^*$ .

#### 4. EXAMPLES

In this final section, we shall present some examples and counterexamples of dynamical systems which contain a dense orbit and have very interesting topological properties. The phase spaces of these dynamical systems will be  $\omega^2 + 1$  and  $\omega^3 + 1$ . The construction of the first example is based in the following combinatorial lemma.

**Lemma 4.1.** *There is a family  $\{A_k : k \in \mathbb{N}\}$  of pairwise disjoint infinite subsets of  $\mathbb{N}$  and an injective function  $h : \bigcup_{k \in \mathbb{N}} A_k \rightarrow \bigcup_{k \in \mathbb{N}} A_k$  such that:*

- (1) *For each  $F \in [A_{2k+2}]^{<\omega}$  there is  $E \in [A_{2k}]^{<\omega}$  such that  $h[A_{2k} \setminus E] \subseteq A_{2k+2} \setminus F$ .*
- (2) *For each  $F \in [A_{2k+1}]^{<\omega}$  there is  $E \in [A_{2k+3}]^{<\omega}$  such that  $h[A_{2k+3} \setminus E] \subseteq A_{2k+1} \setminus F$ .*
- (3) *For each  $E \in [A_0]^{<\omega}$  there are  $H \in [A_1]^{<\omega}$  such that  $h[A_1 \setminus H] \subseteq A_0 \setminus E$ .*
- (4) *There exists  $x_0 \in A_0$  such that  $\mathcal{O}_h(x_0) = \bigcup_{k \in \mathbb{N}} A_k$ .*

*Proof.* To define the function  $h$  and the sets  $A_k$ 's we need the following:

Let  $\{d_n^m : n, m \in \mathbb{N}\}$  be a faithfully enumeration of  $\mathbb{N}$ . Thus we have that  $\{d_n^m : n \in \mathbb{N}\}$  is infinite for each  $n \in \mathbb{N}$ . Define  $a_0 = 0$  and  $a_{n+1} = a_n + 2(n+1) + 1$  for each  $n \in \mathbb{N}$ .

Our point will be  $x_0 = d_0^0$ . Then our main task is to describe the orbit of the point  $d_0^0$  under  $h$ , which is defined by the following rule:

$$d_0^0 \rightarrow d_1^2 \rightarrow d_2^1 \rightarrow d_3^0$$

$$d_3^0 \rightarrow d_4^2 \rightarrow d_5^4 \rightarrow d_6^3 \rightarrow d_7^1 \rightarrow d_8^0$$

$$d_8^0 \rightarrow d_9^2 \rightarrow d_{10}^4 \rightarrow d_{11}^6 \rightarrow d_{12}^5 \rightarrow d_{13}^3 \rightarrow d_{14}^1 \rightarrow d_{15}^0$$

$$d_{15}^0 \rightarrow d_{16}^2 \rightarrow d_{17}^4 \rightarrow d_{18}^6 \rightarrow d_{19}^8 \rightarrow d_{20}^7 \rightarrow d_{21}^5 \rightarrow d_{22}^3 \rightarrow d_{23}^1 \rightarrow d_{24}^0$$

$\vdots$

$$d_{a_n}^0 \rightarrow \cdots d_{a_n+k}^{2k} \rightarrow d_{a_n+k+1}^{2(k+1)} \cdots d_{a_n+n+1}^{2(n+1)} \rightarrow d_{a_n+n+2}^{2n+1} \cdots d_{a_n+2(n+1)-k}^{2k+1} \rightarrow d_{a_n+2(n+1)-(k-1)}^{2(k-1)+1} \cdots d_{a_{n+1}}^0$$

$\vdots$

More precisely,  $h$  is defined as follows:

- (i) For each  $k \in \mathbb{N}$ ,  $n > k - 1$ ,  $h(d_{a_n+k}^{2k}) = d_{a_n+(k+1)}^{2(k+1)}$ .
- (ii) For each  $n \in \mathbb{N}$ ,  $h(d_{a_n+(n+1)}^{2(n+1)}) = d_{a_n+n+2}^{2n+1}$ .

- (iii) For each  $k > 0$ ,  $n \geq k$ ,  $h(d_{a_n+2(n+1)-k}^{2k+1}) = d_{a_n+2(n+1)-(k-1)}^{2(k-1)+1}$ .
- (iv) For each  $n \in \mathbb{N}$ ,  $h(d_{a_n+2(n+1)}^1) = d_{a_{n+1}}^0$ .
- (v) For each  $n \in \mathbb{N}$ ,  $h^{a_n+n+1}(d_0^0) = d_{a_n+n+2}^{2n+1}$ ,  $h^{a_n+k}(d_0^0) = d_{a_n+k}^{2k}$  for all  $0 \leq k < n+1$ , and  $h^{a_n+k}(d_0^0) = d_{a_{n+1}-(a_{n+1}-(a_n+k))}^{2(a_{n+1}-(a_n+k))-1}$  for all  $n+1 < k < a_{n+1} - a_n$ .

Define  $A_0 = \{d_{a_n}^0 : n \in \mathbb{N}\}$  and, for each positive  $k \in \mathbb{N}$ , we define  $A_{2k} = \{d_{a_n+k}^{2k} : n \geq k-1\}$  and  $A_{2k+1} = \{d_{a_n+2(n+1)-k}^{2k+1} : n \geq k\}$ . It is clear that the sets  $A'_k$ s are pairwise disjoint, and also they satisfy the following:

- (a)  $h[A_{2k}] =^* A_{2(k+1)}$  for all  $k \in \mathbb{N}$ .
- (b)  $h[A_{2k+1}] =^* A_{2k-1}$  for all  $k > 0$ .
- (c)  $h[A_1] =^* A_0$ .
- (d)  $h$  is injective.
- (e)  $\mathcal{O}_h(d_0^0) = \bigcup_{m \in \mathbb{N}} A_m$ .

Then we let the reader to verify that (1), (2), (3) and (4) follow directly from (a), (b), (c), (d) and (e).  $\square$

To describe our examples the space  $\omega^2 + 1$  as a subspace of  $\mathbb{R}$  will be written as:

$$\left( \bigcup_{m \in \mathbb{N}} (D_m \cup \{d_m\}) \right) \cup \{d\}.$$

where the points of  $D_m = \{d_n^m : n \in \mathbb{N}\}$  are isolated and form a strictly increasing sequence converging to  $d_m$ , for each  $m \in \mathbb{N}$ , and  $(d_m)_{m \in \mathbb{N}}$  is a strictly increasing sequence converging to  $d$ .

**Example 4.2.** *There is a continuous function  $f : \omega^2 + 1 \rightarrow \omega^2 + 1$  such that*

- (1)  $\mathcal{O}_f(d_m)$  is infinite for all  $m \in \mathbb{N}$ , and
- (2)  $\mathcal{O}_f(d_0^0)$  is dense.
- (3)  $f$  is injective.
- (4)  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$ .

We will use the family  $\{A_m : m \in \mathbb{N}\}$  of pairwise disjoint infinite subsets of  $\mathbb{N}$  and the function  $h$  given in Lemma 4.1. For our convenience, we shall put  $D_m = A_m$  for every  $m \in \mathbb{N}$ . Thus, we define  $f$  on  $\bigcup_{m \in \mathbb{N}} D_m$  by following the function  $h$ . To guarantee the continuity of the function  $f$  the values on the non-isolated points are defined as follows:

- (1)  $f(d_{2k}) = d_{2(k+1)}$  for each  $k \in \mathbb{N}$ .
- (2)  $f(d_{2k+1}) = d_{2k-1}$  for each  $k \in \mathbb{N}$ .
- (3)  $f(d_1) = d_0$ .
- (4)  $f(d) = d$ .

The function  $f$  satisfies the following identities:

- (a)  $f[D_0] = D_2$ .
- (b)  $f[D_{2k}] =^* D_{2(k+1)}$  for every  $k > 0$ .
- (c)  $f[D_{2k+1}] =^* D_{2k-1}$  for every  $k > 0$ .

(d)  $f[D_1] =^* D_0$ .

Let us check that  $f$  is continuous on  $\omega^2 + 1$ . In fact, conditions (a), (b) and (1) guarantee that  $f$  is continuous at  $d_{2k}$  for all  $k$ . Similarly, (c), (d), (2) and (3) show the continuity at  $d_{2k+1}$  for all  $k$ . Finally, to prove the continuity of  $f$  at  $d$ , notice that from (1) and (2) that  $f(d_{n_i}) \rightarrow d$  for every increasing sequence  $(n_i)_{i \in \mathbb{N}}$  of  $\mathbb{N}$ . On the other hand, suppose  $d_{n_i}^{m_i} \rightarrow d$ . Without loss of generality, we may assume that the sequence  $(m_i)_{i \in \mathbb{N}}$  is strictly increasing. By conditions (b) and (c), we conclude that  $f(d_{n_i}^{m_i}) \rightarrow d$ . Therefore,  $f$  is continuous on  $d$ . Now, by condition (4) of lemma 4.1, we obtain that  $\mathcal{O}_f(d_0^0) = \bigcup_{m \in \mathbb{N}} D_m$  and so the orbit of  $d_0^0$  is dense in  $\omega^2 + 1$ . Moreover, we have that  $w_f(x) = \{d_m : m \in \mathbb{N}\} \cup \{d\}$ , for all  $x \in \bigcup_{m \in \mathbb{N}} D_m$ . By Theorem 3.3, we obtain that  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$ , this shows clause (4).  $\square$

Our next example is a dynamical systems  $(\omega^2 + 1, f)$  which is a little different from the previous one.

**Example 4.3.** *There is a continuous function  $f : \omega^2 + 1 \rightarrow \omega^2 + 1$  such that*

- (1)  $\mathcal{O}_f(d_m)$  is finite for all  $m \in \mathbb{N}$ , and
- (2)  $\mathcal{O}_f(d_0^0)$  is dense.
- (3)  $f$  is not injective.
- (4)  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$ .

For our convenience, suppose that  $D_m = \{d_n^m : n \geq m - 1\}$ , for each  $m > 0$ . We define the function  $f : \omega^2 + 1 \rightarrow \omega^2 + 1$  as follows:

- (i)  $f(d_m^0) = d_m^{m+1}$  for every  $m \in \mathbb{N}$ .
- (ii)  $f(d_n^m) = d_n^{m-1}$  for every  $m > 1$  and  $n \geq m - 1$ .
- (iii)  $f(d_m^1) = d_{m+1}^0$  for every  $m \in \mathbb{N}$ .
- (iv)  $f(d_0) = d$ .
- (v)  $f(d_m) = d_{m-1}$  for every  $m > 1$ .
- (vi)  $f(d) = d$ .

From the definition of  $f$  it is clear that

- (a)  $f[D_m] =^* D_{m-1}$  for all  $m > 0$ .
- (b)  $f[D_0] = \{d_m^{m+1} : m \in \mathbb{N}\}$ .

These conditions (a) and (b) guarantee that  $f$  is continuous at  $\omega^2 + 1$ . Applying again Theorem 3.3, we conclude that  $E(\omega^2 + 1, f)$  is homeomorphic to  $\omega^2 + 1$   $\square$

The next counterexample will testify that Theorem 3.3 is false for  $\omega^3 + 1$ . For our convenience, the space  $\omega^3 + 1$  as a subspace of  $\mathbb{R}$  will be written as follows:

The isolated points will be  $D_{i,l} = \{d_{i,l}^k : k \geq l\}$ , for each  $i, l \in \mathbb{N}$  with  $l \geq i$  and

$$\left( \bigcup_{i \in \mathbb{N}, l \geq i} (D_{i,l} \cup \{d_{i,l}\}) \right) \cup \{d_i : i \in \mathbb{N}\} \cup \{d\}.$$

For each  $i \in \mathbb{N}$  and  $l \geq i$ ,  $(d_{i,l}^k)_{l \leq k}$  is a strictly increasing sequence converging to  $d_{i,l}$ ;  $(d_{i,l})_{l \geq i}$  is a strictly increasing sequence converging to  $d_i$ ; and  $(d_i)_{i \in \mathbb{N}}$  is a strictly increasing sequence converging to  $d$ .

**Example 4.4.** *There is a continuous function  $f : \omega^3 + 1 \rightarrow \omega^3 + 1$  such that*

- (1)  $d_i$  is fixed for all  $i \in \mathbb{N}$ .
- (2)  $\mathcal{O}_f(d_{0,0}^0)$  is dense.
- (3)  $f^p$  is discontinuous for all  $p \in \mathbb{N}^*$ .

To define the function  $f$  on the isolated points we start from the point  $d_{0,0}^0$  and then we shall describe its orbit following the rule:

For each  $1 \leq i \leq k$ ,

$$d_{i,i}^k \rightarrow d_{i-1,k}^k,$$

$$d_{0,0}^0 \rightarrow d_{1,1}^1 \rightarrow d_{0,1}^1 \rightarrow d_{0,0}^1$$

$$d_{0,0}^1 \rightarrow d_{2,2}^2 \rightarrow d_{1,2}^2 \rightarrow d_{1,1}^2 \rightarrow d_{0,2}^2 \rightarrow d_{0,1}^2 \rightarrow d_{0,0}^2$$

$$d_{0,0}^2 \rightarrow d_{3,3}^3 \rightarrow d_{2,3}^3 \rightarrow d_{2,2}^3 \rightarrow d_{1,3}^3 \rightarrow d_{1,2}^3 \rightarrow d_{1,1}^3 \rightarrow d_{0,3}^3 \rightarrow d_{0,2}^3 \rightarrow d_{0,1}^3 \rightarrow d_{0,0}^3$$

$\vdots$

$$d_{0,0}^{k-1} \rightarrow d_{k,k}^k \rightarrow d_{k-1,k}^k \rightarrow d_{k-1,k-1}^k \rightarrow \cdots \rightarrow d_{i,k}^k \rightarrow d_{i,k-1}^k \rightarrow \cdots \rightarrow d_{i,i}^k \rightarrow \cdots \rightarrow d_{0,k}^k \rightarrow \cdots \rightarrow d_{0,0}^k$$

$\vdots$

The function  $f$  is defined on the limit and isolated points as follow:

- (1)  $f(d_{i,l}^k) = d_{i,l-1}^k$  for every  $i \in \mathbb{N}$  and  $k \geq l > i$ .
- (2)  $f(d_{i,i}^k) = d_{i-1,k}^k$  for every  $i \in \mathbb{N}$  and  $k > i$ .
- (3)  $f(d_i) = d_i$  for every  $i \in \mathbb{N}$ .
- (4)  $f(d_{i,l}) = d_{i,l-1}$  for every  $i \in \mathbb{N}$  and  $l > i$ .
- (5)  $f(d_{i,i}) = d_{i-1}$  for every  $i > 0$ .
- (6)  $f(d_{0,0}) = d$ .
- (7)  $f(d) = d$ .

It is not difficult to verify that

- (a)  $f[D_{i,l}] = D_{i,l-1}$ , for every  $i \in \mathbb{N}$  and  $l > i + 1$ , and  $f[D_{i,i+1}] = D_{i,i} \setminus \{d_{i,i}^i\}$ .
- (b)  $f[D_{i,i}] = \{d_{i-1,k}^k : k > i\}$  for all  $i > 0$ .
- (c)  $f[D_{0,0}] = \{d_{k,k}^k : k \geq 1\}$ .

First, we will show that  $f$  is continuous on  $\omega^3 + 1$ . Clearly, by identities (6) and (c), we obtain that  $f$  is continuous on  $d_{0,0}$ . Note that conditions (4), (5), (a) and (b) guarantee that  $f$  is continuous at  $d_{i,l}$  for each  $0 < i \leq l$ .

Since  $d_l$  is fixed for each  $l \in \mathbb{N}$ , then (a) implies that  $f$  is continuous at  $d_l$ . It is not hard to verify the continuity of  $f$  at  $d$ .

Finally, let us check that  $f^p$  is discontinuous on  $\omega^3 + 1$ , for each  $p \in \mathbb{N}^*$ . Fix  $p \in \mathbb{N}^*$  and  $1 \leq i$ . We know that  $(d_{i,l})_{i < l}$  is a strictly increasing sequence converging to  $d_i$ . By condition (4) and (5), for each  $i < l$  there exist  $n_l \in \mathbb{N}$  such that  $f^m(d_{i,l}) = d_{i-1}$  for  $m \geq n_l$ . Consequently,  $f^p(d_{i,l}) = d_{i-1}$  for each  $l < i$ , but  $d_i$  is fixed and so  $f^p(d_i) = d_i$ . Therefore,  $f^p$  is discontinuous at  $d_i$ .  $\square$

We finish with a list of open questions that the authors were not able to solve.

The functions given in Examples 4.2, 4.3 and 4.4 have a dense orbit. It is then natural to ask:

**Question 4.5.** *Giving a countable ordinal  $\alpha > 3$ , is it possible to define a continuous function  $f : \omega^\alpha + 1 \rightarrow \omega^\alpha + 1$  with dense orbit?*

The following question is related to Theorem 3.3 and Example 4.4.

**Question 4.6.** *Given a discrete dynamical system  $(\omega^\alpha + 1, f)$  with dense orbit, where  $\alpha \geq 3$  is a countable ordinal, is  $E(\omega^\alpha + 1, f)$  always countable?*

It is not hard to see that  $E(\omega^\alpha + 1, f)$  could be a convergent sequence with its limits point without regarding the size of  $\alpha \geq 1$ . This leads to formulate the next question.

**Question 4.7.** *Let  $\alpha, \beta > 3$  be countable ordinals. Is there a continuous function  $f : \omega^\alpha + 1 \rightarrow \omega^\alpha + 1$  such that  $E(\omega^\alpha + 1, f)$  is homeomorphic to  $\omega^\beta + 1$ ?*

The previous question may be stated in a more general form:

**Question 4.8.** *Given a compact metric countable space  $X$ , is there a continuous function  $f : X \rightarrow X$  such that  $E(X, f)$  is homeomorphic to  $X$ ?*

According to Theorem 2.7, if  $E(X, f)$  is countable, then  $P_f$  must be finite. Hence, we may ask:

**Question 4.9.** *Given a compact metric countable dynamical system  $(X, f)$ , if  $P_f$  is finite, must  $E(X, f)$  be countable?*

Examples 4.2 and 4.3 provide two different functions  $f_1, f_2 : \omega^2 + 1 \rightarrow \omega^2 + 1$  such that  $E(\omega^2 + 1, f_i)$  is homeomorphic to  $\omega^2 + 1$  for  $i = 1, 2$ . The remarks following the proof of Theorem 3.3 assert that  $E(\omega^2 + 1, f_i)$ , for  $i = 1, 2$ , is not algebraically isomorphic to  $\omega^2 + 1$  equipped with ordinal addition. These facts naturally suggests the next question that could be easy to answer.

**Question 4.10.** *Are the Ellis semigroups constructed in 4.2 and 4.3 algebraically homeomorphic?*

We know, by Theorem 2.7, that if  $P_f$  is infinite and  $X$  is countable, then  $E(X, f)$  is homeomorphic to  $2^{\mathbb{N}}$ . But what about the algebraic operation:

**Question 4.11.** *Suppose  $X$  is compact metric countable space and  $f, g : X \rightarrow X$  are continuous functions such that  $P_f$  and  $P_g$  are infinite. Are the Ellis semigroups  $E(X, f)$  and  $E(X, g)$  algebraically homeomorphic?*

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